

Lecture 2 (Feb 1, 2016)

Phase plane analysis (second order systems)

$\dot{x} = f(x)$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$: a vector field on \mathbb{R}^2

$$(*) \quad \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), & x_1(0) = x_1^0 \\ \dot{x}_2 &= f_2(x_1, x_2), & x_2(0) = x_2^0 \end{aligned}$$

Note. We only consider autonomous systems because we want to study phase plane with vector fields on it and these vector fields shouldn't change with time.

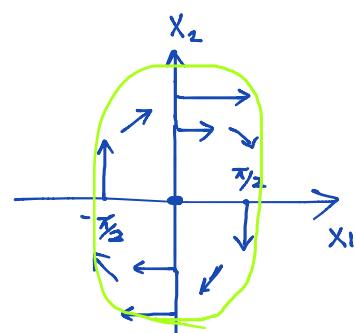
Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of $(*)$:

- $x_1 - x_2$ plane is called "phase plane."
- The curve $x(t)$ for all $t > 0$ in the phase plane is called the "trajectory (orbit)" of $(*)$ from $x(0)$.
- one can draw the "phase portrait" (family of all trajectories) without solving the differential equation. For any point x draw arrow for vector $f(x)$.

Example. (Pendulum with no friction)

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix}$$

Matlab command: "quiver"



origin is an equilibrium

Linear systems

$$\dot{x} = Ax, A \in \mathbb{R}^{2 \times 2}$$

Let λ_1, λ_2 be eigenvalues of A . There exists $T \in \mathbb{R}^{2 \times 2}$ nonsingular s.t. $J = T^{-1}AT$, J is real Jordan form of A :

- a) If $\lambda_1, \lambda_2 \in \mathbb{R}$ & $\lambda_1 \neq \lambda_2$ then $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- b) If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ then $J = \begin{pmatrix} \lambda & k \\ 0 & \lambda \end{pmatrix}$ $k=1$ or $k=0$
- c) If $\lambda_1, \lambda_2 = \alpha \pm \beta i$ then $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

Note that origin is an equilibrium point.

If $\lambda_i=0$, then origin is not an isolated equilibrium.

Consider $\dot{x} = Ax$, $x(0) = x_0$. Then

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{TJT^{-1}t}x_0 = Te^{Jt}T^{-1}x_0 \\ e^{TJT^{-1}t} &= 1 + TJT^{-1}t + \frac{1}{2}(TJT^{-1}t)^2 + \dots \\ &= TT^{-1} + T(Jt)T^{-1} + \frac{t^2}{2} T J \cancel{\left(T^{-1}\right)} J T^{-1} + \dots \\ &= T \left(1 + tJ + \frac{t^2 J^2}{2} + \dots \right) T^{-1} \\ &= T e^{Jt} T^{-1} \end{aligned}$$

change of coordinates: $z = T^{-1}x$. Then

$$\dot{z} = T^{-1}\dot{x} = T^{-1}Ax = T^{-1}ATz = Jz \Rightarrow \dot{z} = e^{Jt}z_0, z_0 = T^{-1}x_0$$

a) If $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq 0, \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$ then

$$Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = e^{\begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix}} z_0 = \begin{pmatrix} e^{\lambda_1 t} z_1^0 \\ e^{\lambda_2 t} z_2^0 \end{pmatrix}$$

$$\text{i.e. } z_1(t) = e^{\lambda_1 t} z_1(0), \quad z_2(t) = e^{\lambda_2 t} z_2(0)$$

Eliminating t between the two equations, we obtain

$$(*) \quad z_2 = c z_1^{\frac{\lambda_2}{\lambda_1}} \quad \text{where } c = z_2(0)/z_1(0)^{\frac{\lambda_2}{\lambda_1}}$$

The phase portrait of the system is given by the family of curves generated from (*) by allowing c to take arbitrary values in \mathbb{R} (arbitrary initial conditions).

Suppose $\lambda_2 < \lambda_1 < 0$

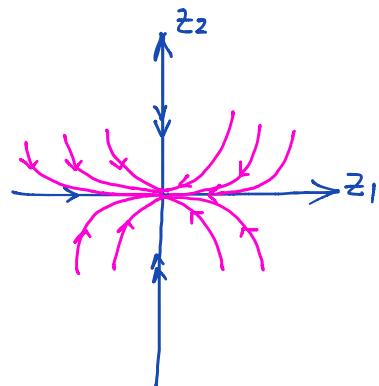
In z_1 - z_2 coordinates:

λ_2 : fast eigenvalue

λ_1 : slow



z_2 decays faster than z_1



In original $(x_1 - x_2)$ coordinates:

Let v_1 & v_2 be eigenvectors in original coordinates : $T = [v_1 \ v_2]$

$$x = Tz \Rightarrow x(t) = v_1 z_1(t) + v_2 z_2(t)$$

$$= e^{\lambda_1 t} z_1(0) v_1 + e^{\lambda_2 t} z_2(0) v_2$$

\Rightarrow behavior on z_1 axis now along v_1

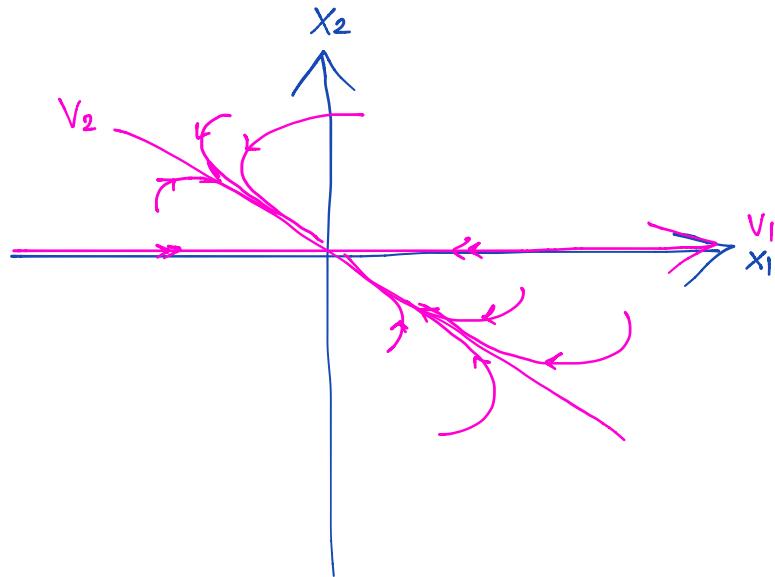
z_2

v_2

Example: $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = -\frac{1}{4}x_2 \end{cases} \Rightarrow A = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{pmatrix}$

eigenvalues: $\lambda_1 = -1, \lambda_2 = -\frac{1}{4} \quad \lambda_1 < \lambda_2 < 0$

eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$



If $\lambda_1 < \lambda_2 < 0$ or $\lambda_2 < \lambda_1 < 0$, $x=0$ is called an "stable node".

If $\lambda_1 > \lambda_2 > 0$, $x=0$ is called an "unstable node".

In example, if

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = \frac{1}{4}x_2$$

then $\lambda_1 = 1, \lambda_2 = \frac{1}{4}$.

Same picture as above but arrows go in other direction.

Now if $\lambda_1 < 0 < \lambda_2$, then $x=0$ is a "saddle point."

stable unstable
eigenvalue =

eig. vect. v_1 v_2

Trajectories have hyperbolic shapes (except for eigenvectors)

Example. $\dot{x}_1 = 3x_1 - 2x_2$

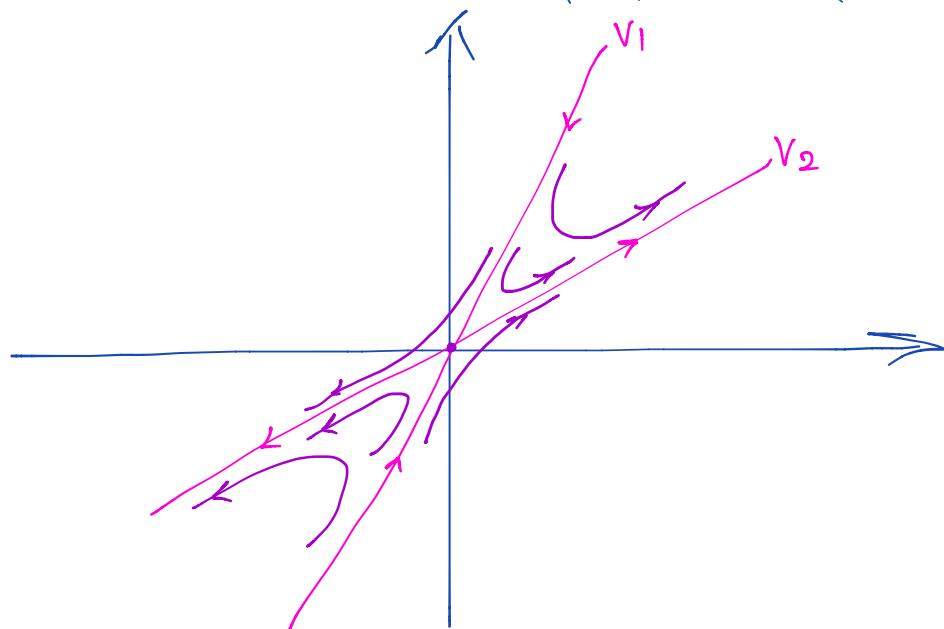
$$\dot{x}_2 = 2x_1 - 2x_2$$

$$\Rightarrow \lambda_1 = -1$$

$$v_1 = \begin{pmatrix} .45 \\ .89 \end{pmatrix}$$

$$\lambda_2 = 2$$

$$v_2 = \begin{pmatrix} .89 \\ .45 \end{pmatrix}$$



b) non-zero repeated eigenvalues

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, $\lambda \neq 0$ then $k=1$ or $k=0$ and

$$e^{\lambda t} = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} kt \\ 0 & e^{\lambda t} \end{pmatrix} \Rightarrow z_1(t) = e^{\lambda t} z_1(0) + kte^{\lambda t} z_2(0)$$

$$z_2(t) = e^{\lambda t} z_2(0)$$

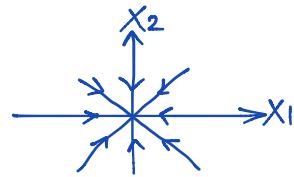
Example

$$\dot{x}_1 = -x_1$$

$$\dot{x}_2 = -x_2$$

$$A = J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, k=0, \lambda=-1$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



stable node

no asymptotic fast or slow

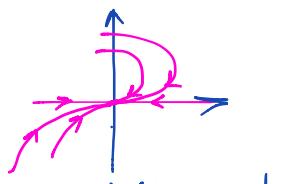
Example

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = -x_2$$

$$A = J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, k=1, \lambda=-1$$

$$v_1 = v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



stable node

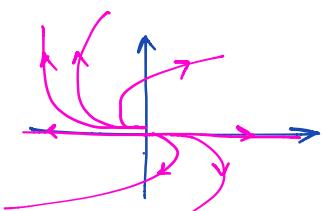
Example.

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = x_2$$

$$A = J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, k=1, \lambda=1$$

$$v_1 = v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



unstable node

c) complex eigenvalues

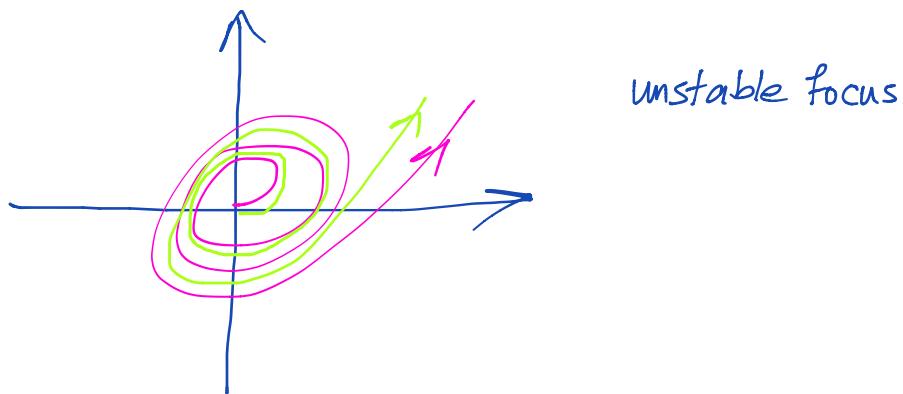
$$\lambda_{1,2} = \alpha \pm i\beta$$

“stable focus” if $\alpha < 0$

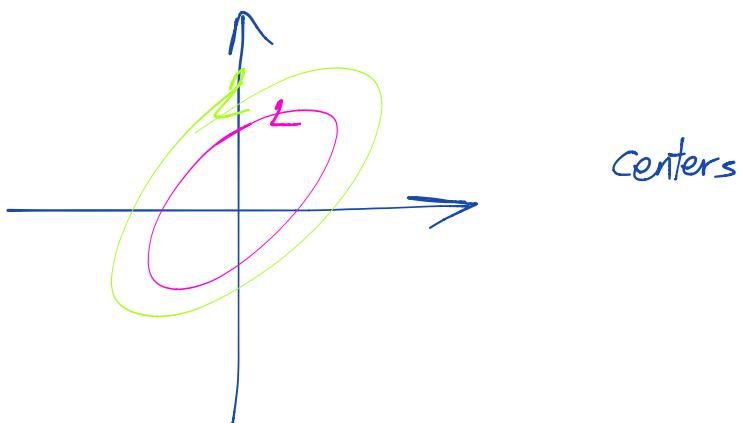
“unstable focus” if $\alpha > 0$

“Center” if $\alpha = 0$

Example. $\dot{x}_1 = 2x_1 - 2.5x_2$
 $\dot{x}_2 = 1.8x_1 - x_2$ $\Rightarrow \lambda_{1,2} = .5 \pm 1.5i$



Example. $\dot{x}_1 = 2x_1 - 5x_2$
 $\dot{x}_2 = x_1 - 2x_2$ $\Rightarrow \lambda_{1,2} = \pm i$

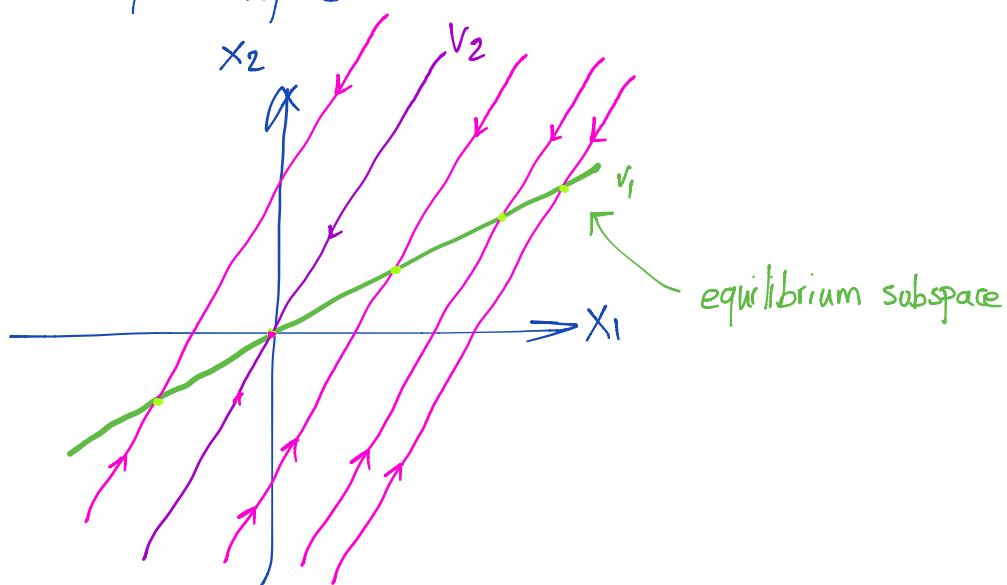


d) zero eigenvalues (special degenerate case)

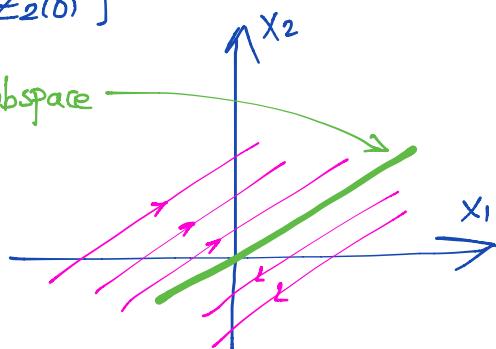
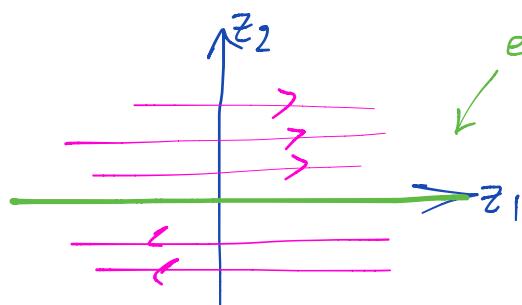
□ 1 dim nullspace. Any point in nullspace is equilibrium point.

$$J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow z_1(t) = z_1(0), z_2(t) = e^{\lambda_2 t} z_2(0) \quad \lambda_2 < 0$$

v_1 spans nullspace.



$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 0 \end{aligned} \Rightarrow z_2(t) = z_2(0) \quad \left. \begin{aligned} z_1(t) &= z_1(0) + z_2(0)t \\ &= z_1(0) + z_2(0)t \end{aligned} \right\} \Rightarrow z_1(t) = z_2(0)t + z_1(0)$$



□ 2 dim nullspace.

$$J = 0 \Rightarrow z(t) = z(0) \quad \text{every point in the plane is an equilibrium.}$$

